Chapter 1 TRANSFORMATIONS OF FUNCTIONS

This sample contains the *first page only* of each section in the workbook.

Two complete chapter samples, including all practice questions, are also provided, for:
1 - Transformations
7 - Trigonometric Identities and Equations

1.1 Prerequisite Skills + Translations of Functions

Part A – Prerequisite Skills

Review of Two Familiar Functions

In this units we'll take some known and new functions and apply various transformations. And that means, if you're eager with anticipation, to alter the function's equation or graph.

However before we get into all of that – over the next few pages (and 6 warm-ups), we'll brush up on some key concepts we'll need in this first unit and throughout this course. Starting with – some functions from Math 20!

Warm-up **Exploration #1**

The Quadratic Function – The Graph of $y = x^2$

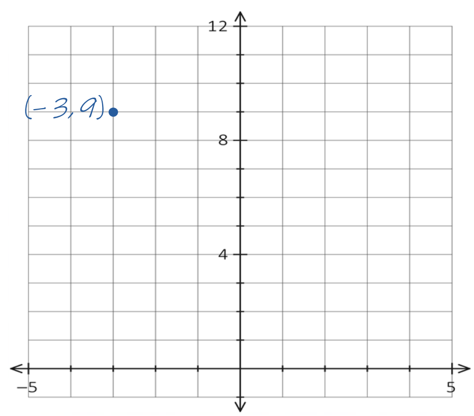
1 ➡ **Complete** the **table of values** on the right, and **plot** the points to **sketch** the graph.

2 ➡ **State** the **domain** and **range** of the function.

Domain ______ Range ______

3 ➡ On the same grid, **sketch the graph** of $y = x^2 + 3$. *Add 3 to all y-coordinates, verify on your graphing calc.*

x	$y = x^2$
-3	$(-3)^2 = 9$
-2	
-1	
0	
1	
2	
3	



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The Absolute Value Function – the Graph of $y = |x|$

4 ➡ **Complete** the **table of values** on the right, and **plot** the points to **sketch** the graph.

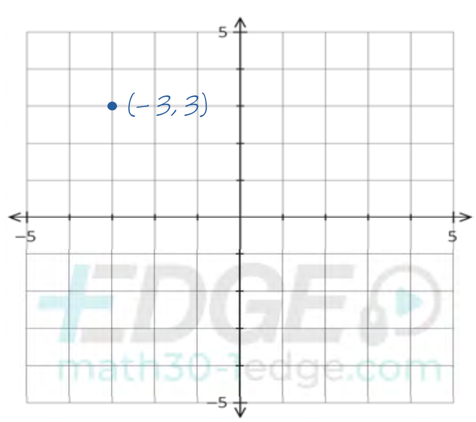
5 ➡ **State** the **domain** and **range** of the function.

Domain ______ Range ______

6 ➡ On the same grid, **sketch the graph** of $y = |x| - 2$.

Explain how the graph compares to $y = |x|$.

x	$y = |x|$
-3	$|-3| = 3$
-2	
-1	
0	
1	
2	
3	



1.2 Reflections

In the last section we looked at **translations** – in which the position of a graph is altered. *Think – picking up a graph and moving it left / right or up / down …. and then dropping it.*

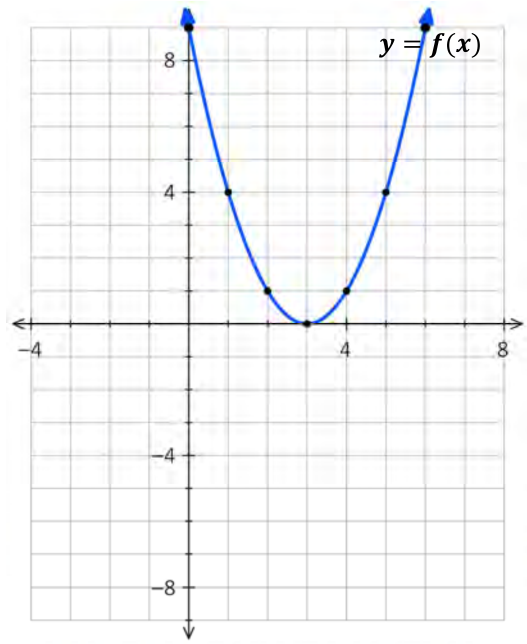
We'll next consider **reflections**, which affects a graphs orientation. *Think of – lifting the top or bottom (or left side or right side) of a graph and flipping it over. (Vertically or horizontally)*

Warm-up Exploration #1

Vertical Reflections

1 ➡ The graph on the function $f(x) = (x-3)^2$ is on the right. **Sketch** the mirror image of the graph, reflected in the x-axis. *Be sure to indicate the placement of the 7 indicated points.*

Label the new graph $y = g(x)$.



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2 ➡ **Describe** how the coordinates change from the graph of $y = f(x)$ to $y = g(x)$.

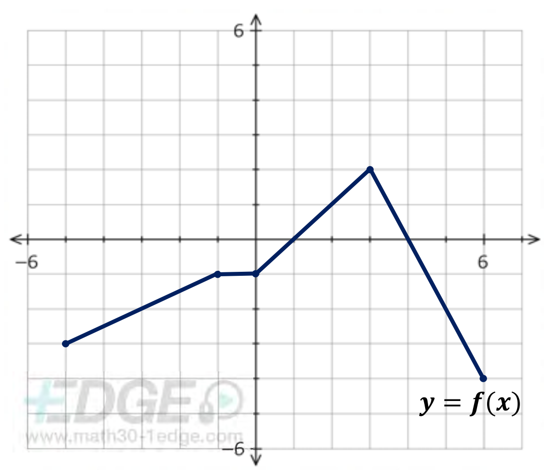
3 ➡ **State** the equation, in terms of $y = f(x)$.

4 ➡ **Explain** how to change the equation of a function, such as $y = (x-3)^2$, so that the graph is reflected about the x-axis. Determine an equation of the reflected graph, in terms of x.

Exploration #2

1 ➡ The graph of $y = f(x)$ is on the right. **Sketch** the mirror image graph, again reflected in the x-axis.

Label the new graph $y = g(x)$, and label the new coordinates of the five indicated points.



2 ➡ **State** the mapping rule that describes the change in all coordinates from $y = f(x)$ to $y = g(x)$.

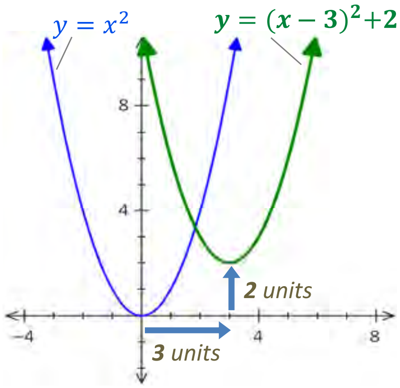
3 ➡ **State** the coordinates of the two points that are on both the graph of $y = f(x)$ and $y = g(x)$. What do they have in common?

Exploration #3

1 ➡ Sketch function $y_1 = x^2 - 4x + 5$ in your graphing calculator. State the coordinates of the vertex.

2 ➡ Sketch function $y_2 = -(x^2 - 4x + 5)$ in your graphing calculator and state the coordinates of the vertex. How do the two graphs compare?

Once again – let's first look back where we've been….



First, we saw how **translations** occur when we *add (or subtract) numbers.*

For example, a function $y = x^2$ can be transformed to $y = (x-3)^2+2$

Here we subtract 3 from $\boldsymbol{x}$	And we subtract 2 from $\boldsymbol{y}$*
Graph horizontally translates 3 units right	***And vertically translates 2 units up***
(opposite direction of the sign)	(same direction* of the sign)

Another way we can look at this is to apply *replacements:*

- To ***horizontally translate*** the graph 3 units right, replace "x" with "$x-3$"
- To ***vertically translate*** the graph 2 units up, replace "y" with "$y-2$"

So, $\boldsymbol{y = x^2}$ *becomes* $y - 2 = (x-3)^2$ → *Simplifies to* $\boldsymbol{y = (x-3)^2 + 2}$

When we think of vertical translations this way, we can treat it "the same" as horizontal!

**That is, the opposite direction of the sign in the equation:* $y = f(x) \rightarrow \boldsymbol{y - k = f(x - h)}$

We can similarly think of **reflections** with replacements.

For example, given the graph of $y = (x-4)^2$ …

- To ***horizontally reflect***, replace "x" with "$-x$"

  $\boldsymbol{y = (-x - 4)^2}$

  We can optionally simplify to:

  $y = [-1(x+4)]^2$

  $y = (-1)^2(x+4)^2$

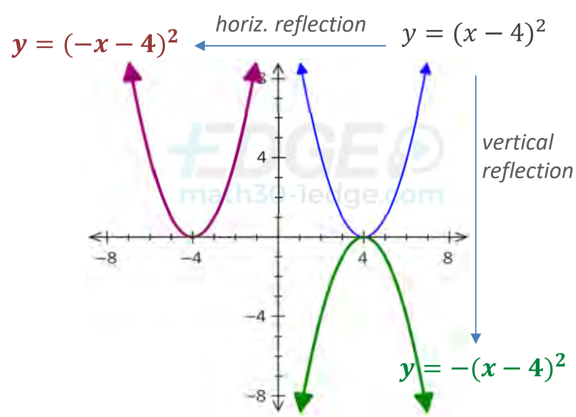
  → $y = (x+4)^2$

- To ***vertically reflect***, replace "y" with "$-y$"

  $\boldsymbol{-y = (x - 4)^2}$

  Now divide both sides by -1 *to isolate "y"!*

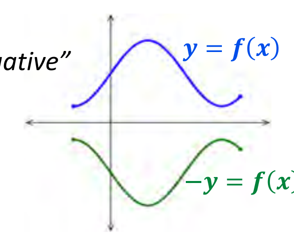
  → $y = -(x-4)^2$



Replacing "y" with "−y" is identical to "making the entire right side negative"

$$-y = f(x) \Leftrightarrow y = -f(x)$$

→ Vertical reflection about the x-axis

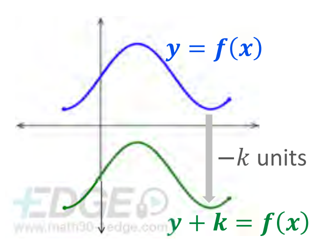


And replacing "y" with "y − k" is identical to "adding k to the function"

$$y - k = f(x) \Leftrightarrow y = f(x) + k$$

→ Vertical translation k units

- up, if $k > 0$ - down, if $k < 0$



1.4 Combining Transformations

In the previous section we considered problems that involved both a stretch and reflection.

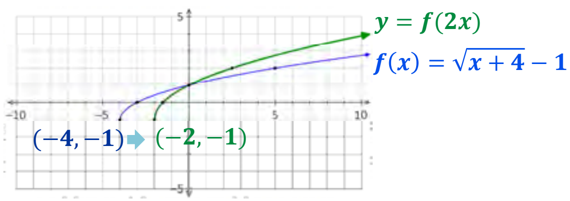
And we found that the order in which we applied the transformations didn't matter.

For example, consider the graph of $f(x) = \sqrt{x+4} - 1$

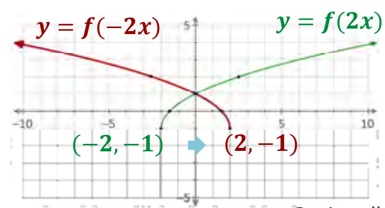
Suppose we wish to apply a horizontal stretch about the y-axis by a factor of $1/2$ and apply a horizontal reflection about the y-axis.

- We can apply the ***stretch first***, and then the reflection….

Horizontal stretch by a factor of 1/2



And then horizontal reflection

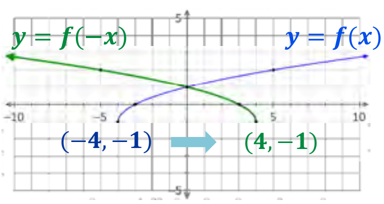


Optionally simplify

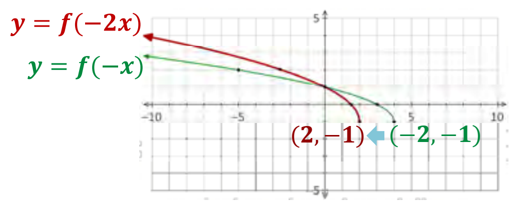
Equation in terms of x: $y = \sqrt{2x+4} - 1$ → $y = \sqrt{-2x+4} - 1$ $\quad y = \sqrt{-2(x-2)} - 1$

- Or we can apply the ***reflection first***, and then the stretch ….

Horizontal reflection



And then the horizontal stretch, factor 1/2

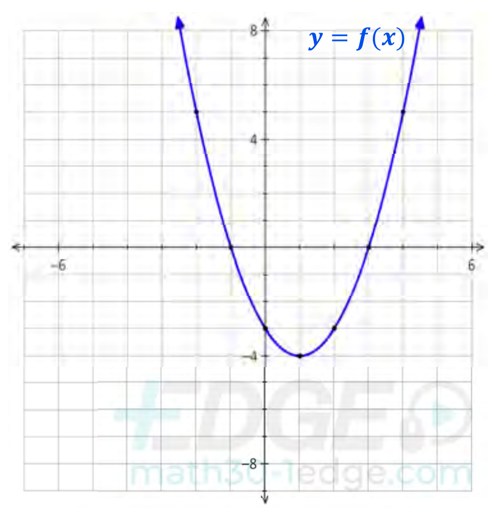


Equation in terms of x: $y = \sqrt{-x+4} - 1$ → $y = \sqrt{-2x+4} - 1$ *Same equation! (and same graph)*

warmup

Warm-up **Exploration #1** **Combining a vertical stretch** (or reflection) **with a horizontal translation**

The graph of $f(x) = (x-1)^2 - 4$ is shown below



1 On the same grid, construct a new graph of $y = g(x)$ by first applying a vertical stretch about the x-axis by a factor of 2, then applying a horizontal translation 4 units left.

2 Determine an equation of $y = g(x)$, in terms of x, by applying the transformations in opposite order:

- First apply a horizontal translation 4 units left.

- Then apply a vertical stretch by a factor of 2.

3 Does the equation developed in #2 match the graph made in #1? Is the order in which a vertical stretch and horizontal translation are applied relevant?

1.5 Inverse of a Relation

You can think of the **inverse** as "undoing", or more specifically – doing the opposite operations in the opposite order.

For example, the inverse of walking into a room and turning on the lights is to turn off the lights and then leave the room.

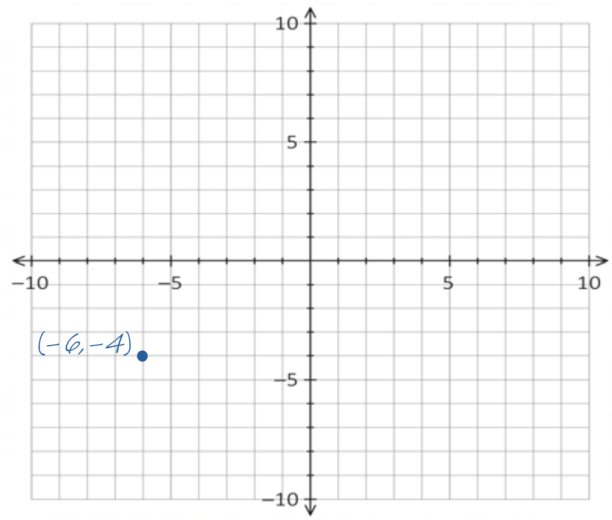
Warm-up
Exploration #1

Sketching the Graph of an Inverse

Consider the function $f(x) = \frac{1}{3}x - 2$

1 ➡ **Complete** the first column of the table below by substituting the given values of x into the equation for $f(x)$.

x	$y = f(x)$	$y = g(x)$
-6	$\frac{1}{3}(-6)-2=-4$ ➔ $(-6,-4)$	
-3		
0		
3		
6		
9		



2 ➡ Plot each of the points on the grid on the right to **sketch** the graph of $f(x)$. *The first point is plotted for you.*

3 ➡ **Complete** the $g(x)$ column by interchanging all of the $f(x)$ coordinates. *Note that the first point in the column will be (–4,–6).*

4 ➡ Plot each of the points in the $g(x)$ column to **sketch** the graph of $y = g(x)$ on the same grid.

5 ➡ **Sketch** the graph of $y = x$, also on the same grid.

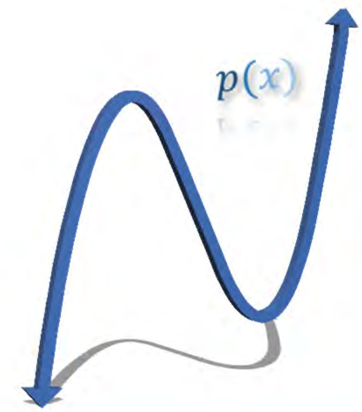
6 ➡ **Compare** the distances from the line $y = x$ of points on the graph of $f(x)$ and corresponding points on the graph of $g(x)$.

7 ➡ Use terminology from this unit to **describe** the transformation of the graph of $y = f(x)$ to the graph of $y = g(x)$. Where are the **invariant points** in this transformation?

8 ➡ Are the graphs of $y = f(x)$ and $y = g(x)$ functions? Explain.

9 ➡ Determine an **equation** for $y = g(x)$. How does this equation relate to the equation for $y = f(x)$?

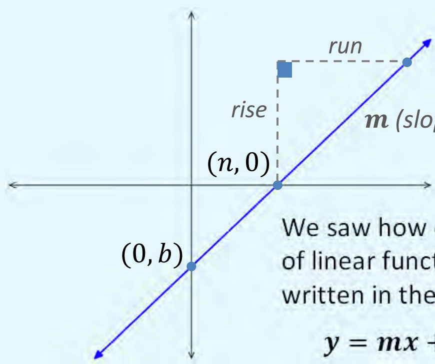
Chapter 2 POLYNOMIAL FUNCTIONS



2.1 Characteristics of Polynomial Functions

You've already studied Polynomial functions! *Remember these?*

In Math 10C you studied **Linear Functions**



m *(slope)* $=$ ***rise*** / ***run***

We saw how equations of linear functions can be written in the form

$$y = mx + b$$

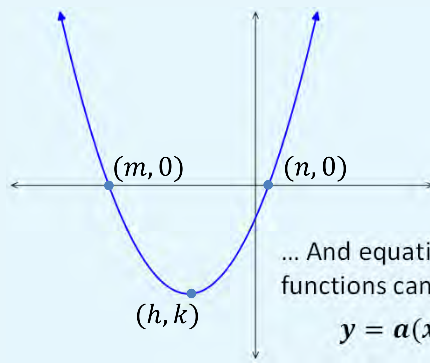
*Where m is the **slope** of the line, and b is the **y-intercept***

Note that the linear functions can also be written in the form $y = m(x - n)$

*Where n is the **x-intercept***

➡ These are **degree 1** Polynomial Functions

And in Math 20-1 you studied **Quadratic Functions**



… And equations of quadratic functions can be written

$$y = a(x - h)^2 + k$$

*Where a is the **vertical stretch**, and the coordinates of the **vertex** are (h, k).*

Note that the quadratic functions can also be written in the form $y = a(x - m)(x - n)$

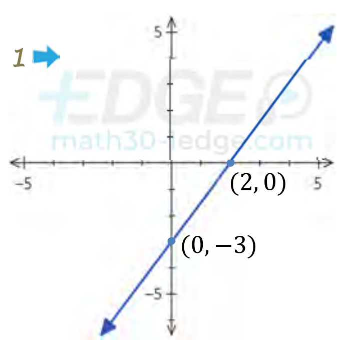
*Where m, n are **x-intercepts***

➡ These are **degree 2** Polynomial Functions

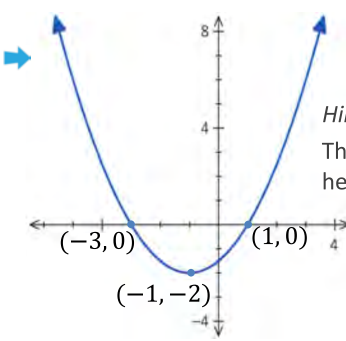
warm up

Determine an equation for each of the following functions:

1 ➡



2 ➡



Hint:

The vertical stretch here, a, is $1/2$

2.2 Dividing Polynomials and the Remainder Theorem

Zeros of a Polynomial Functions

Warm-Up #1

Each of the polynomial functions below is written in both expanded and factored form.

Expanded Form	Factored Form
$f(x) = 6x - 12$	$f(x) = 6(x-2)$
$g(x) = x^2 - 2x - 8$	$g(x) = (x+2)(x-4)$
$p(x) = x^3 - 7x + 6$	$p(x) = (x+3)(x-1)(x-2)$

1 ➡ State the **zeros** (values of x for which the function equals zero) beside the factored form of each function above.

2 ➡ State the relationship between the factored form of a polynomial equation, and the zeros / x-intercepts of its graph.

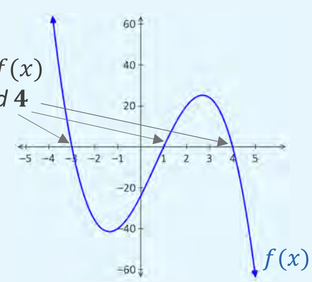
3 ➡ Convert the following 2nd degree polynomial functions to factored form, to determine the zeros / x-intercepts of the graphs: (a) $g(x) = x^2 - x - 6$ (b) $f(x) = 2x^2 + 7x - 4$

4 ➡ Suppose we wish to factor a 3rd degree function $p(x) = x^3 - 6x^2 + 11x - 6$.
We are told an equivalent; partially factored form is: $p(x) = (x-1)(x^2 - 5x - 6)$
Determine the fully factored form of $p(x)$,
and state the **zeros** of the function.

The **zero** of a function is the value of x for which the function equals zero.
This is represented graphically by the **x-intercepts**.

So if a function $f(x)$ has zeros of $x = -3$, 1 and 4, then the graph will have x-intercepts at $(-3, 0)$, $(1, 0)$ and $(4, 0)$.

The zeros of $f(x)$ are $-3, 1$ and 4



The zeros of a function ⟷ *Correspond to* ***The x-intercepts on the graph***

When a polynomial function is in **factored form**, the zeros of the function can be easily identified.

For example, our function here is $f(x) = -2(x+3)(x-1)(x-4)$.

Zeros are: -3 1 4

Bringing it all together…..

A useful tool in determining characteristics of polynomial functions (in this case, zeros) – is **factoring**.

In prior courses we factored a lot of second degree (quadratic) polynomial functions, such as $g(x) = x^2 - x - 6$ and $f(x) = 2x^2 + 7x - 4$

So, what we now need is a method to factor 3rd (or higher) degree polynomials, so that we can algebraically determine the zeros of a function such as $p(x) = x^3 - 6x^2 + 11x - 6$.

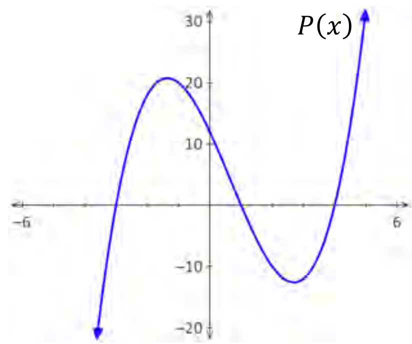
And this method involves **dividing polynomials**, which is what we'll look at in this section.

2.3 The Factor Theorem

warm up

The graph of $P(x) = x^3 - 2x^2 - 11x + 12$ is shown on the right. The graph has integer x-intercepts.

The factored form of $P(x)$ is
$P(x) = (x - a_1)(x - a_2)(x - a_3)$;
where a_1, a_2, and a_3 are zeros of the function.



1 ➡ State the factored form of $P(x)$. Explain how the graph can be used to determine the factored form.

2 ➡ Find 3 remainders; for when $P(x)$ is divided by each of its three factors as stated above.

3 ➡ State the relationship between the factors of a polynomial expression, the zeros of the corresponding polynomial function, and the remainder theorem.

Recall that the **remainder theorem** states that when a polynomial, $P(x)$, is divided by a binomial in the form $x - a$, the remainder is $\boldsymbol{P(a)}$.

The **factor theorem** states that $x - a$ is a factor of a polynomial function $P(x)$, if $\boldsymbol{P(a) = 0}$. (That is, if dividing by a factor gives no remainder)

Worked Example Use the factor theorem to show that the polynomial function $P(x) = x^3 + 5x^2 + 3x - 9$ has factors of $(x + 3)$ and $(x - 1)$.

Solution: The factor theorem states that if $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$. *(That is, there is no remainder)*

- Test $(x + 3)$ *... is it a factor?*

  (x + 3) is a factor if P(−3) = 0

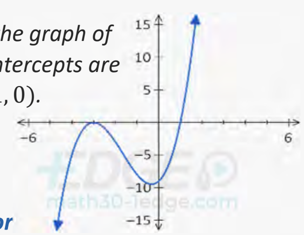
  $P(-3) = (-3)^3 + 5(-3)^2 + 3(-3) - 9$
  $= -27 + 45 - 9 - 9$
  $= \mathbf{0}$ ⬅ *No remainder ...* $(x + 3)$ ***is a factor***

- Test $(x - 1)$ *... is it a factor?*

  (x − 1) is a factor if P(1) = 0

  $P(1) = (1)^3 + 5(1)^2 + 3(1) - 9$
  $= 1 + 5 + 3 - 9$
  $= \mathbf{0}$ ⬅ *No remainder ...* $(x - 1)$ ***is a factor***

Note: *Here's the graph of* $P(x)$. *The x-intercepts are* $(-3, 0)$ *and* $(1, 0)$.



Class Example 2.31 *Using the Factor Theorem to test given Binomials*

A polynomial function is given as $\boldsymbol{P(x) = 3x^4 - 4x^3 - 19x^2 + 8x + 12}$. Use the factor theorem to show that:

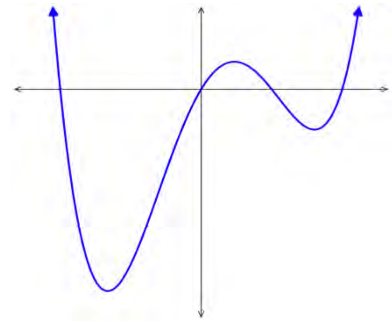
(a) $(x - 1)$ is a factor of $P(x)$ (b) $(3x + 2)$ is a factor of $P(x)$ (c) $(x + 1)$ is a NOT a factor

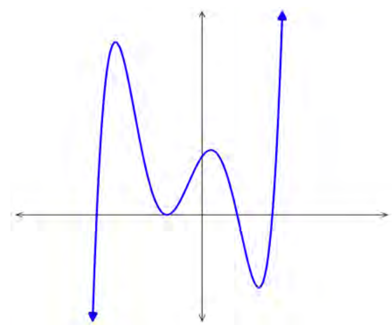
2.4 Further Analysis of Polynomial Function Graphs

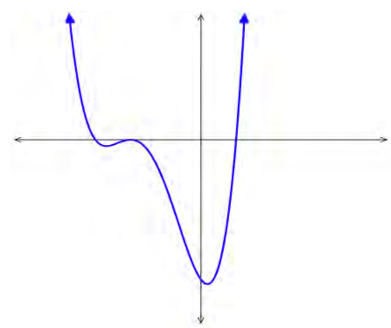
Warm-Up #1

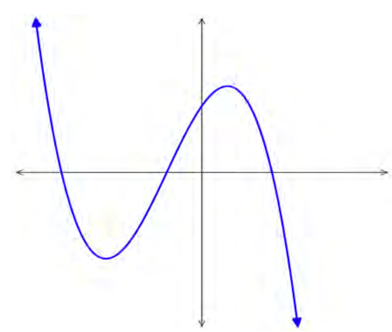
Labeled A to E below are five separate functions, provided in both expanded equivalent and factored form.

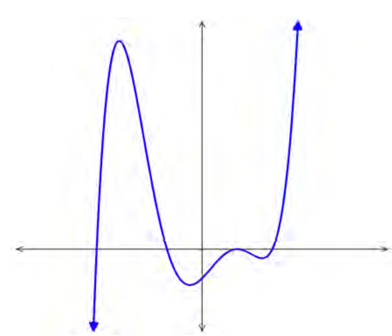
1 ➡ For each function below, *without using your graphing calculator,* match with the appropriate graph on the right, and determine each of the indicated graph characteristics.











A

Expanded form: $y = -2x^3 - 6x^2 + 12x + 16$

Factored form: $y = -2(x+1)(x-2)(x+4)$

Degree: ______ y-intercept: ________ Max / min?: ________

x-intercepts: _______________________

B

Expanded form: $y = x^4 + 6x^3 + 9x^2 - 4x - 12$

Factored form: $y = (x+2)^2(x-1)(x+3)$

Degree: ______ y-intercept: ________ Max / min?: ________

x-intercepts: _______________________

C

Expanded form: $y = \frac{1}{2}x^4 - x^3 - 8x^2 + 16x$

Factored form: $y = \frac{1}{2}x(x+4)(x-2)(x-4)$

Degree: ______ y-intercept: ________ Max / min?: ________

x-intercepts: _______________________

D

Expanded form: $y = x^5 - 8x^3 + 6x^2 + 7x - 6$

Factored form: $y = (x+3)\,(x+1)(x-1)^2(x-2)$

Degree: ______ y-intercept: ________ Max / min?: ________

x-intercepts: _______________________

E

Expanded form: $y = x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6$

Factored form: $y = (x+3)\,(x+1)^2(x-1)(x-2)$

Degree: ______ y-intercept: ________ Max / min?: ________

x-intercepts: _______________________

2 ➡ Functions B, D, and E each have a degree 2 factor, such as such as $(x+2)^2$ for function B.

Analyze and explain the effect on the graph, when a factor is squared.

Chapter 3 EXPONENTIAL and LOGARITHMIC FUNCTIONS

3.1 Exponential Expressions and Equations

Background Skills – Exponent Rules

warm up

To successfully complete this unit, and even enjoy it, we must first brush up our sills on a concept last thoroughly visited in Math 10C – **exponents**.

Exponent

$3^4 = 3 \times 3 \times 3 \times 3$

Base   *Appears 4 times*

Exponent Rules *(Remember these?)*

Visit **math30-1edge.com** for solutions to all warm-ups and class examples

Name	Rule	Example (simplify each)
Product of Powers	$(b^m)(b^n) = b^{m+n}$	$x^3 \times x^4 =$
Quotient of Powers	$\frac{b^m}{b^n} = b^{m-n}$	$\frac{x^5}{x^3} =$
Power of a Power	$(b^m)^n = b^{mn}$	$(y^3)^2 =$
Power of a Product	$(ab)^m = a^m b^m$	$(4x^4)^2 =$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{2x}{y^2}\right)^3 =$
Zero Exponent	$b^0 = 1$	$(7y^2 \times 2y^6)^0 =$
Negative Exponents	$b^{-n} = \frac{1}{b^n}$ or $\frac{1}{b^{-n}} = b^n$	$\frac{1}{2^{-4}} =$
Neg. Exp. *fraction base*	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{2}{3}\right)^{-2} =$
Rational Exponents	$b^{\frac{m}{n}} = \sqrt[n]{b^m}$ or $(\sqrt[n]{b})^m$	$8^{\frac{2}{3}} =$

Answers are at the bottom of the next page

Try Each First!

Don't peek!

Evaluate each of these three, try w/o using your calc!

3.2 Graphs of Exponential Functions

We saw in the previous section how exponential equations involved terms where the variable is in the exponent. **Exponential functions** are can be used to model many real-world situations.

- *The world population*
- *The value of an investment earning positive interest*
- *The measured amount of a decaying radioactive isotope*
- *The value of a used car*
- *The temperature of a cup of hot chocolate as it cools*

Exponential functions can model any of these, given certain parameters.

We'll dive further into applications in section 4.7, after learning a bit about logarithms. For now we'll focus on the basic graphs.

*These three involve **negative exponential growth**, or **exponential decay** ➔*

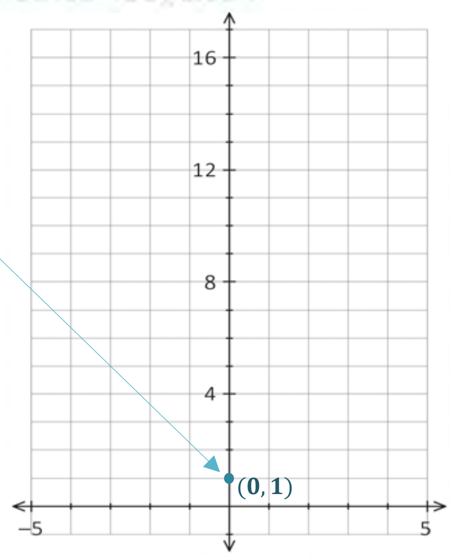
Exploration #1 The Graph of $y = 2^x$

Visit math30-1edge.com for solutions to all warm-ups and class examples

warmup

1 ➡ Complete the table of values below, and plot the remaining points on the graph. Then, sketch the smooth curve that goes through each of your points. Then – you will have graphed your first exponential function.

x	2^x
0	$2^0 = \mathbf{1}$
1	
2	
3	
4	
-1	
-2	
-3	



2 ➡ ***Function Essentials:***

State each of the following

Domain

Range

Asymptote

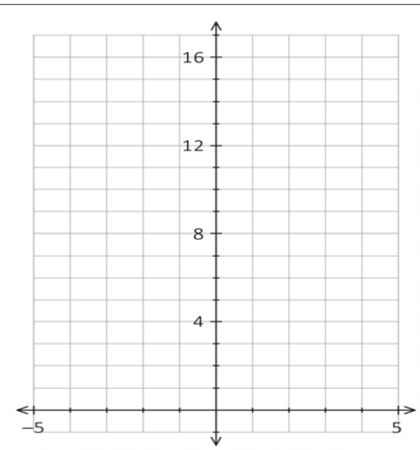
x-intercept

y-intercept

The Graph of $y = (\frac{1}{2})^x$

Next, we sketch the graph of the function obtained by **horizontally reflecting** the graph of $y = 2^x$, about the line $x = 0$.

3 ➡ Use transformations to show that the resulting function equation is: $y = \left(\frac{1}{2}\right)^x$



4 ➡ ***Function Essentials:***

Are there any differences from the graph of $y = 2^x$?

3.3 The Logarithmic Function

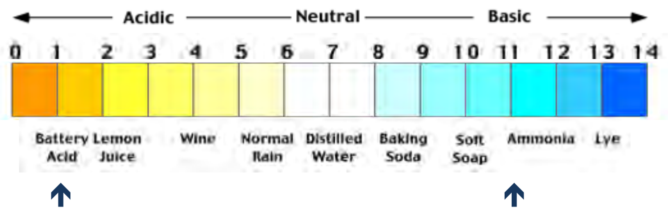
What exactly are *logarithms*?

Logarithms can be helpful in putting very large (or very small) numbers on a *human-friendly scale.*

For example, on the pH scale →

Ammonia, which has a pH of 12, is ***11 higher*** than battery acid, which has a pH of 1…

But Ammonia is ***one hundred billion*** times as alkaline!



Every increase of 1 on the pH scale means a tenfold increase in the alkalinity of a substance. (Similarly, every decrease of one means a tenfold increase in the acidity) The pH scale (and others – such as the Richter scale, or Decibel scale), are examples of ***logarithmic scales***. *Teaser – more on those in section 4.8!*

Orders of magnitude

The difference between large numbers can be hard to comprehend.

A "trick" we can use is to write these numbers in terms of **inputs**, that, is, as a power base 10.

10 000	*Ten Thousand* →	4	*Since* $10^4 = 10\,000$
1 000 000	*One Million* →	6	*Since* $10^6 = 1$ *million*
1 000 000 000 000	*One Trillion* →	12	*Since* $10^{12} = 1$ *trillion*
1 000 000 000 000 000	*One Quadrillion* →	15	*Since* $10^{15} = 1$ *Quadrillion*

These inputs are called logarithms.

IE *- The logarithm, base 10, of one million is 6*

Defining the Logarithmic Function as an Inverse

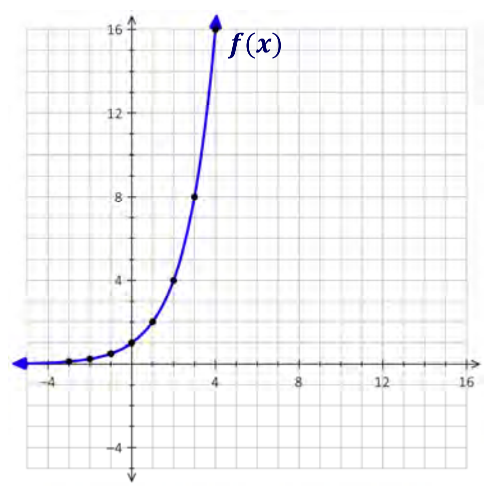
Pt. on graph of $y = 2^x$	Corresponding pt. on graph of inverse
$(-3, \quad)$	
$(-2, \quad)$	
$(-1, \quad)$	
$(0, \quad)$	
$(1, \quad)$	
$(2, \quad)$	
$(3, \quad)$	
$(4, \quad)$	

warm up **Exploration #1**

The graph of $f(x) = 2^x$ is shown:

1 ➡ Use the graph to complete the ← table on the left.

2 ➡ Sketch the graph of the inverse on the same grid on the right →



4 ➡ Determine the equation of the inverse function

3 ➡ Indicate the domain and range of each function:

	$f(x)$	**Inverse**
D:		
R:		

3.4 Logarithm Laws

In section 4.1 we reviewed the exponent rules. A corresponding concept is the Laws of Logarithms. *Remember – the value of a logarithm is an exponent, after all!*

warm up

We start by considering the sum of two logarithms of the same base, such as: $log_2 8 + log_2 4$

Is there a simple way to evaluate this? See if you can spot the patterns below.

1 $\boldsymbol{log_2 8 + log_2 4}$

$2^{\square} = 8\,?$ $\quad 2^{\square} = 4\,?$

$= \underline{3} + \underline{2}$

$= \underline{5}$ *Copy answer here*

Now find x: $log_2 x = \underline{5}$

$x = \underline{32}$

So..... $\boldsymbol{log_2 8 + log_2 4} = \underline{log_2 32}$

2 $\boldsymbol{log_3 9 + log_3 27}$

$3^{\square} = 9\,?$ $\quad 3^{\square} = 27\,?$

$= \_\_\_\_ + \_\_\_\_$

$= \_\_\_\_$ *Copy answer here*

Now find x: $log_3 x = \_\_\_\_$

$x = \_\_\_\_$

So..... $\boldsymbol{log_3 9 + log_3 27} = \_\_\_\_\_\_\_\_$

3 $\boldsymbol{log 100\,000 - log 1000}$

$10^{\square} = 100\,000\,?$ $\quad 10^{\square} = 1000\,?$

$= \_\_\_\_ - \_\_\_\_$

$= \_\_\_\_$ *Copy answer here*

Now find x: $log x = \_\_\_\_$

$x = \_\_\_\_$

So..... $\boldsymbol{log 100\,000 - log 1000} = \_\_\_\_\_\_\_\_$

---

4 $\boldsymbol{log_2 8^2} = log_2(\_\_\_)$

$= \_\_\_\_$

Now evaluate this:

$= 2log_2 8$

$= 2\,(\_\_\_) = \_\_\_\_$ *So.....* $\boldsymbol{log_2 8^2} = \_\_\_\_\_\_\_\_$

The LOGARITHM LAWS:

❶ $log_b(M \times N) = log_b M + log_b N$

❷ $log_b\left(\frac{M}{N}\right) = log_b M - log_b N$

❸ $log_b(M)^n = n\,log_b M$

❹ $log_b(c) = \frac{log_a c}{log_a b}$

Change of base identity, introduced in 3.3

Example:

$\underbrace{log(1\,000\,000)}_{=6} = \underbrace{log_b 1000}_{=3} + \underbrace{log_b 1000}_{3}$

$\underbrace{log_2(4)}_{=2} = log_2\left(\frac{64}{16}\right) = \underbrace{log_2 64}_{=6} - \underbrace{log_2 16}_{4}$

$\underbrace{log_3 729}_{=6} = log_3(9^3) = 3\underbrace{log_3 9}_{=2} = 3(2) = \boldsymbol{6}$

$\underbrace{log_{16} 256}_{=2} = \underbrace{\frac{log_4 256}{log_4 16}}_{=\frac{4}{2}} = \underbrace{\frac{log_2 256}{log_2 16}}_{=\frac{8}{4}} = \underbrace{\frac{log 256}{log 16}}_{\text{Verify on calc!}}$

3.5 Solving Exponential Equations using Logs, and Applications

Earlier we saw how some exponential equations can be solved by *re-writing terms in the same base*.
This method works great - when terms *can be* written in the same base: → $2(5)^{x-1}=50$

warm up

1 → **Algebraically solve** the following equation, by re-writing in the same base.

$$2(5)^{x-1}=50$$

2 → **Show** that we cannot as easily solve the following equation using the same method

$$2(5)^{x-1}=60$$

Solving any Exponential Equation

Only some equations can be solved using the method we learned in 4.1, namely re-writing in the same base.
In order to solve any exponential equation we need a broader method – and that method involves **logarithms**.
(bet you saw that coming)

To solve any exponential equation (and not just those special cases where we can equate the bases), we can either:

solve: $\mathbf{2(5)^{x-1}=60}$

Method 1:

Convert to logarithmic form

$$\frac{\cancel{2}(5)^{x-1}}{\cancel{2}}=\frac{60}{2}$$ *First, isolate the power term*

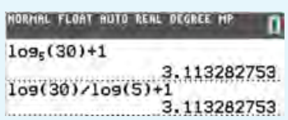
$$(5)^{x-1}=30$$ *Convert to log form*

$$\log_5(30)=x-1$$ *Isolate x*

$$\boxed{x=\log_5(30)+1}$$ *Exact solution*

$$\boxed{x\approx 3.11}$$ *Approx. solution*

Use calculator for approx. solutions →



Method 2:

Or, take the logarithm of both sides

$$(5)^{x-1}=30$$ *Isolate power term*

$$\mathbf{log}(5)^{x-1}=\mathbf{log}30$$ *"log" both sides*

*… and isolate **x** using the power log law*

$$(\mathbf{x-1})\log 5=\log 30$$

$$x-1=\frac{\log 30}{\log 5} \Rightarrow \boxed{x=\frac{\log 30}{\log 5}+1}$$

Exact solution ↑
(same as $\mathbf{\log_5(30)+1}$)

Class Example 3.51 *Solving an exponential equation using logarithms*

Use an algebraic process to solve each of the following equations. *State solutions correct to the nearest hundredth.*

(a) $(3)2^{5-4x}=300$

(b) $2(5)^{x-2}=152$

3.6 Solving Logarithmic Equations and Log Scales

1 – Solving Logarithmic Equations

warm up

You may recall we solved some logarithmic equations already in section 4.3. Let's see what you remember, try solving each of the following:

1 ➡ Solve: $log_{\frac{1}{3}}(x) = -2$

2 ➡ Solve: $log_{16}(x-1) = \frac{3}{4}$

For each of these simple equations, finding the solution involves *converting to exponential form*. This is a common method for solving logarithmic equation, so let's call it a ***type 1 logarithmic equation***.

Now, as we're want to do in Math 30-1, we're going to kick things up a notch. Let's consider logarithmic equations that involve first applying laws of logarithms or some other simplification.

3 ➡ **Simplify the left side of the following equation using log laws. Then, algebraically solve:** $log_2 x - log_2 5 = 3$

Granted, we didn't kick to too high a notch there, but we soon will! But first let's look at a type of logarithmic equation where all terms involve logarithms. For these, which we'll call ***type 2 logarithmic equations***, we do not convert to exponential form. Instead, we use log laws to combine one side of the equation to a single log, then solve by log-dropping.

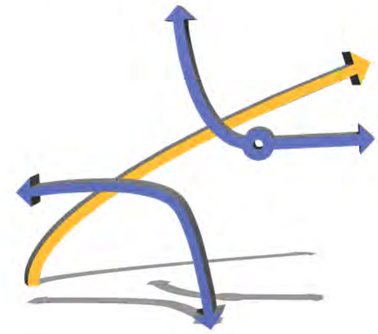
4 ➡ **Simplify the left side of the following equation using log laws. Then, algebraically solve:** $log_3 x + log_3 4 = log_3 24$

5 ➡ **Verify your solution to #4 by substituting your answer back into the original equation.**

6 ➡ **Graphically solve the equation from #4 using your calculator.**

Chapter 4

RADICAL FUNCTIONS, RATIONAL FUNCTIONS



4.1 The Radical Function

warm up

As we encountered in Chapter 1, the basic radical function is $f(x) = \sqrt{x}$.

In the first part of this chapter, we'll get re-acquainted with this function, and further examine some graph characteristics.

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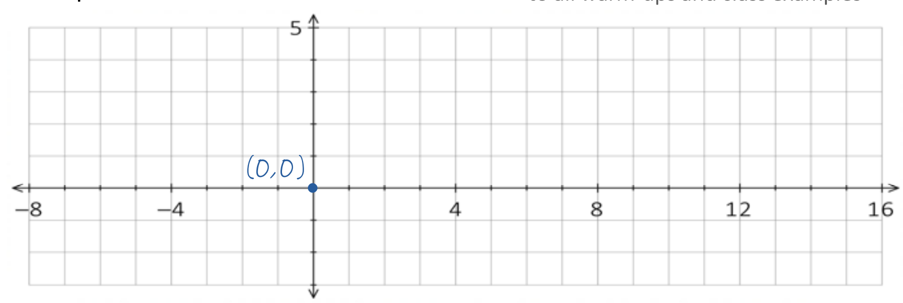
1 ➡ Complete the **table of values** below

x	$f(x) = \sqrt{x}$
0	$\sqrt{0} = 0$
1	
4	
9	
16	
−1	



2 ➡ Plot the points above to **sketch** the graph of $f(x) = \sqrt{x}$. Label the graph ①.

3 ➡ **State** the **domain** and **range** of the function. ________ Domain ________ Range

4 ➡ Use transformations to **sketch** the graph of $g(x) = \sqrt{-(x-9)}$ on the same grid above. Label it ②.

$(0, 0) \rightarrow$

$(1, 1) \rightarrow$

$(4, 2) \rightarrow$

$(9, 3) \rightarrow$

$(16, 4) \rightarrow$

5 ➡ Algebraically determine the **domain** of $y = g(x)$, by setting what's under the square root sign *greater than or equal to zero*.

6 ➡ Algebraically confirm the **y-intercept** of $y = g(x)$, by setting the value of x to zero and evaluating.

7 ➡ Algebraically confirm the **x-intercept** of $y = g(x)$, by setting the value of y to zero and solving.

warm up

Consider the graphs of $f(x) = x - 2$ and $y = g(x)$ below.

In this warm-up you'll be asked to consider the **value** of the functions for some particular x. As a refresher, in this context: *value* = *y-coordinate*.

For example, the value of $f(x)$ at $x = -1$, that is $f(-1)$, is -3. *On the graph of $y = f(x)$ the y-coordinate when x-coordinate is -1 is -3. We can express this as $\boldsymbol{f(-1) = -3}$.*

1 Compare the values of the two functions shown, at the following values of x.

$f(18) =$ ____ $g(18) =$ ____

$f(11) =$ ____ $g(11) =$ ____

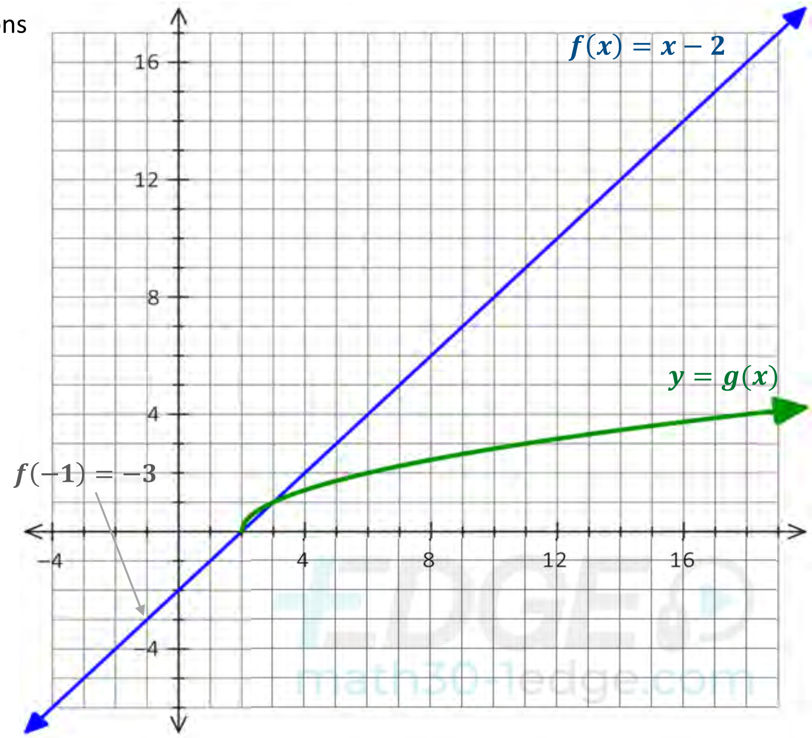
$f(6) =$ ____ $g(6) =$ ____

$f(3) =$ ____ $g(3) =$ ____

$f(2) =$ ____ $g(2) =$ ____

$f(1) =$ ____ $g(1) =$ ____

$f(0) =$ ____ $g(0) =$ ____



2 Describe how the values of $g(x)$ compare to those of $f(x)$ for any x.

3 Complete a mapping rule which describes the transformation of all points on the graph of $f(x)$ to the graph of $g(x)$.

4 Based on your answer above, provide an equation for $g(x)$, in terms of x. Verify using your graphing calculator.

5 State the domain and range of each function.

____	____	____	____
$f(x)$ domain	$f(x)$ range	$g(x)$ domain	$g(x)$ range

6 Explain how the *graph* of $f(x)$ relates to the *domain* of $g(x)$.

7 Describe the location of the invariant points as the graph of $f(x)$ transforms to the graph of $g(x)$.

warm up

Warm-Up #1 The Graph of $y = \frac{1}{x}$

1➡ **Complete** the (partial) **table of values** on the left. *We started for you! Four of the points are already plotted.*

2➡ **Sketch** $y = 1/x$ on your graphing calculator, using the window shown.

x: $[-5, 5, 1]$ y: $[-5, 5, 1]$

min max scl

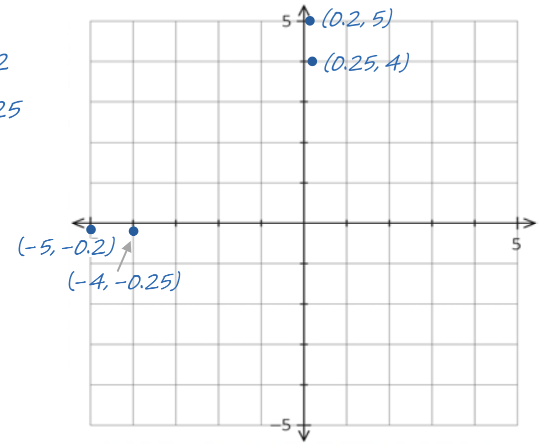
3➡ With the help of you calculator, **sketch** the graph by connecting the plotted points in a smooth curve.

4➡ *Fill in the blanks:* The graph of $y = \frac{1}{x}$

has a vertical asymptote (V.A.) at _____

and a horizontal asymptote (H.A.) at ______

x	$y = 1/x$
-5	$1/-5 = -0.2$
-4	$1/-4 = -0.25$
-3	
-2	
-1	
-0.5	
-0.25	
-0.2	
0	
0.2	$1/0.2 = 5$
0.25	$1/0.25 = 4$
0.5	
1	
2	
3	
4	
5	



5➡ The domain of the function is:

____

And the range is:

____

Warm-Up #2 The Graph of $y = \frac{a}{x}$

1➡ Use transformations to sketch the graph of $y = \frac{2}{x}$

The mapping rule is: $(x, y) \rightarrow$ ______

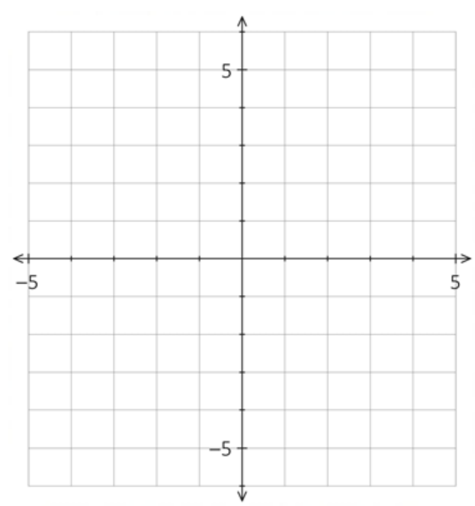
2➡ *Fill in the blanks:* The graph of $y = \frac{2}{x}$

has a vertical asymptote (V.A.) at _____

and a horizontal asymptote (H.A.) at ______

3➡ The domain of the function is: _______________

And the range is: ________________

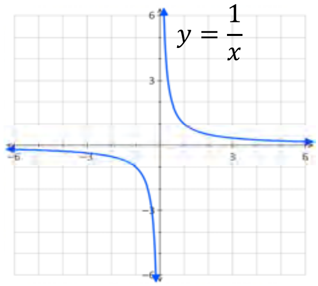


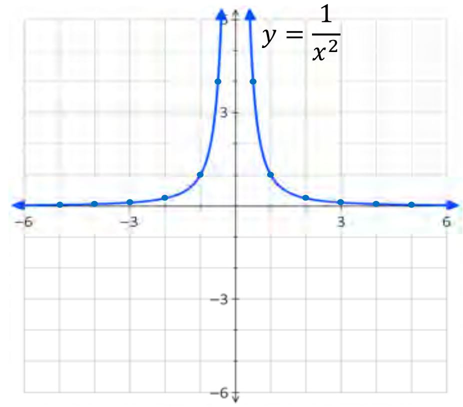
warmup

Exploration #1 The Graph of $y = \frac{1}{x^2}$

The graph of $y = \frac{1}{x^2}$ and accompanying table of values are shown.

1 **Compare** the graph of $y = \frac{1}{x^2}$ to that of $y = \frac{1}{x}$, shown below.





x	$y = 1/x^2$
-5	$1/(-5)^2 = 0.04$
-4	$1/(-4)^2 = 0.0625$
-3	$1/(-3)^2 = 0.111...$
-2	$1/(-2)^2 = 0.25$
-1	$1/(-1)^2 = 1$
-0.5	$1/(-0.5)^2 = 4$
-0.25	$1/(-0.25)^2 = 16$
-0.2	$1/(-0.2)^2 = 25$
0	undefined
0.2	$1/(0.2)^2 = 25$
0.25	$1/(0.25)^2 = 16$
0.5	$1/(0.5)^2 = 4$
1	$1/(1)^2 = 1$
2	$1/(2)^2 = 0.25$
3	$1/(3)^2 = 0.111...$
4	$1/(4)^2 = 0.0625$
5	$1/(5)^2 = 0.04$

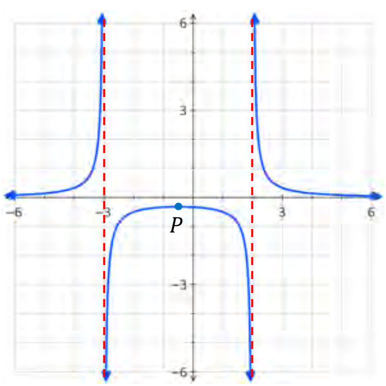
2 **Describe** the following characteristics of the graph of $y = \frac{1}{x^2}$:

Domain	*Range*	*V.A.s*	*H.A.*	*x -int*	*y -int*

Exploration #2

Refer to the graphs below to answer to the following questions. For the graph on the left, point P is a local maximum of the first graph, with coordinates $(-0.5, -0.32)$.

$$y = \frac{2}{(x+3)(x-2)}$$



Note: Point P is a *local maximum*, with coordinates $(-0.5, -0.32)$.

V.A.s:

H.A.:

Domain:

Range:

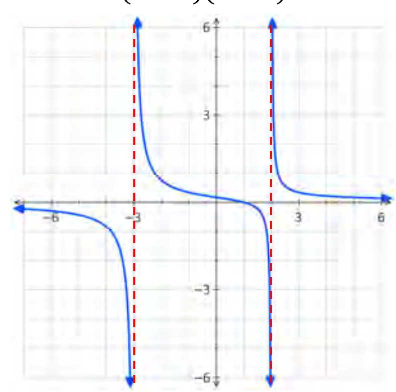
x-int:

y-int:

Degree of numerator:

Degree of denominator:

$$y = \frac{x-1}{(x+3)(x-2)}$$



V.A.s:

H.A.:

Domain:

Range:

x-int:

y-int:

Degree of numerator:

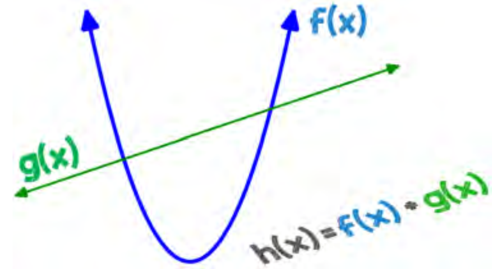
Degree of denominator:

1 **Explain** how each characteristic relates to specific parts of the equation.

x-intercept: *Vertical Asymptote:* *Horizontal Asymptote:*

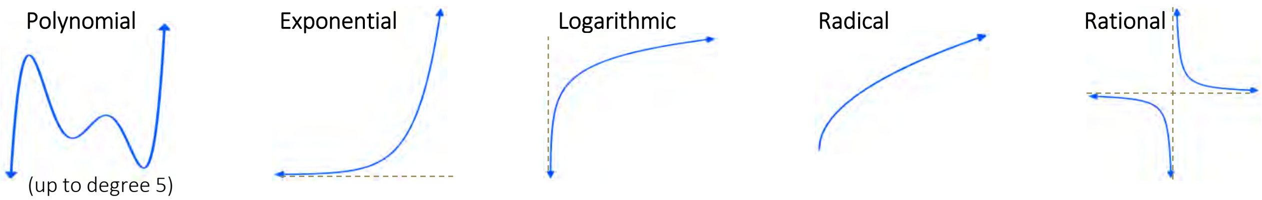
2 Based on your analysis for the graph on the right – can a rational function graph cross its horizontal asymptote? .

Chapter 5 OPERATIONS on FUNCTIONS

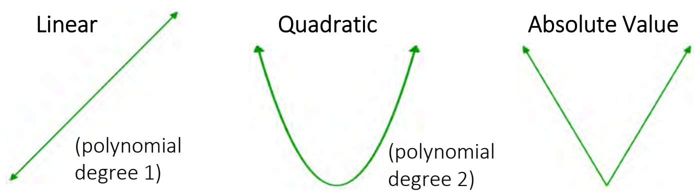


5.1 Adding, Subtracting, Multiplying and Dividing Functions

Functions – in our first four chapters we got in-depth on so many types of functions and their graphs!



These along with functions we studied in prior courses: (these were reviewed in chapter 1)



In this chapter we'll apply various types of operations on these functions as well as other functions that could be defined as just a set of ordered pairs, a graph, or a table. It will be a good review of were we've been so far, while tying together some of the core function concepts studied earlier. *Let's get started….*

Combining Functions

1 ➡ Use the graph of the three functions on the right to **complete** the table below

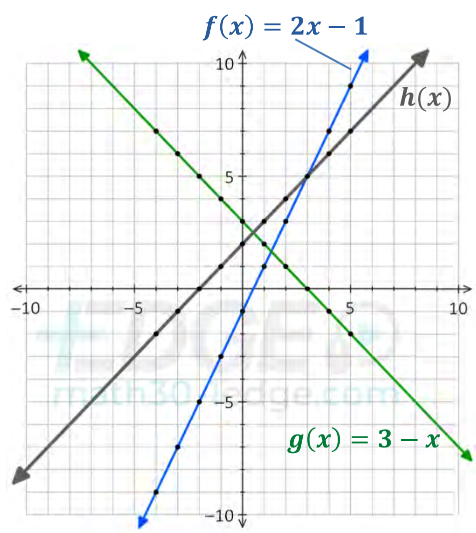
x	$f(x)$	$g(x)$	$h(x)$
-4	-9	7	-2
-3			
-2			
-1			
0			
1			
2			
3			
4			
5			

2 ➡ Examine the table to determine the relationship between the values in each column.

3 ➡ State an equation for $h(x)$ in terms of $f(x)$ and $g(x)$.

4 ➡ State an equation for $f(x)$ in terms of $h(x)$ and $g(x)$.

5 ➡ Determine a slope-intercept form equation for $h(x)$.



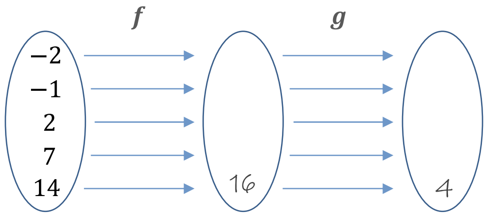
6 ➡ Add the function equations for $f(x)$ and $g(x)$ to show that $h(x)$ is the sum.

Exploration – Composite Functions

warmup

Suppose we define two function functions: $f(x) = x + 2$ and $g(x) = \sqrt{x}$.

1 ➡ Complete the following mapping diagram, such that the output values from f become the input values in g: (To help get things going – one example is done for you, as is #2)

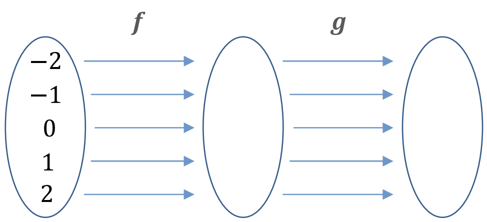


2 ➡ Write a function, $h(x)$, that obtains the result from the first step directly.

$h(x) = \sqrt{x+2}$

Next, let's make those functions: $f(x) = 3x$ and $g(x) = x^2 + 1$

3 ➡ Complete the following mapping diagram, such that the output values from f become the input values in g:



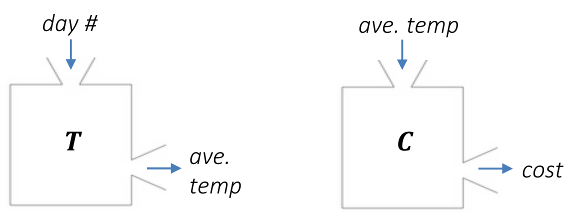
4 ➡ Write a function, $h(x)$, that obtains the result from the first step directly.
Test by substituting some of start points. Is the same result obtained?

Consider this example: Suppose we wish to derive a function that will determine the cost to heat a house on a particular day of the year. The cost to heat the house will depend on the daily average temperature, which in turn depends on the particular day of the year.

We can think of this as involving two functions:

- A **temperature** function, where the input is the day number, and the output the daily average temperature…
- … which then becomes the input in a **cost** function, where the input is average temperature.

Expressing this visually using the function input / output machine model, we have:



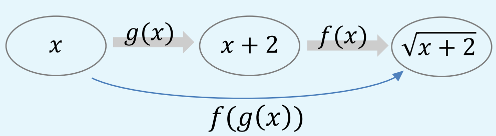
The cost of heating a home, as a function of day number in the year is:

$\boldsymbol{C(T(d))}$

By combining these two relationships into one function, we have performed **function composition**.

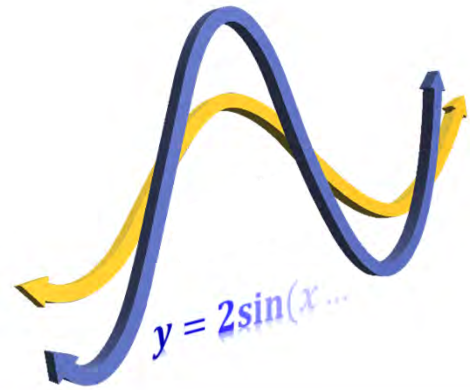
A **composite function** $f(g(x))$ is a function that is composed of one function inside another.

For example, consider the functions from the first warm-up:



- Another way we write $f(g(x))$ is $(f \circ g)(x)$
  $(f \circ g)(x)$ is read: "f of g of x"
- Order matters: $(f \circ g)(x)$ is not the same as $(g \circ f)(x)$
- Substitute the expression for $g(x)$ into $f(x)$ to determine the composite function $(f \circ g)(x)$

Chapter 6 TRIGONOMETRIC FUNCTIONS



6.1 Radian Measure and Arc Length

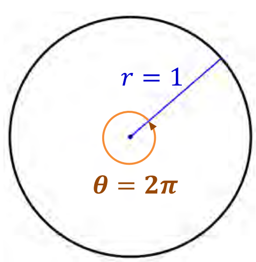
Radians – A new Way to Measure Angles

So far in your Trig Career, you've dealt with problems where angles are measured in **degrees**. And it's served you well!

We define one full rotation as 360°, which likely comes from our ancestor's observation the motion of the sun and stars followed patterns on a 365-day circle.

For simplicity, they decided to round to 360, which is a good thing, as it's a *highly composite number*.
(360 is divisible by 180, 90, 60, 45, 30, etc)

That said, most of the mathematics and scientific communities use a different angular measure - **radians**.



Consider a circle with a radius of one.

The circumference of this circle is:

$C = \pi \times d$ ➔ $C = 2\pi r$ ➔ $C = 2\pi(1)$

$\boldsymbol{C = 2\pi}$

This 2π value is also the radian measure of the angle θ, representing one full rotation.

note

Recall that π is the ratio of circumference of a circle to its diameter

$$\pi = \frac{circumference}{diameter}$$

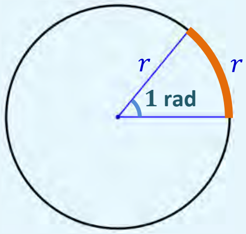
π is an irrational number, it's decimal form can only be approximated!

3.14159265358979238....

One rotation is 2π in radians. $360° = 2\pi$

We need not include "radians" or "rads" as a unit – any angle measure given without a degree symbol is assumed to be in radians!

One radian is the measure of the angle formed by rotating the radius of a circle through an arc equal in length to the radius.



One radian is approximately equal to $\mathbf{57°}$ ← $180° \div \pi$

Since one rotation is $2\pi = 360°$, dividing both sides by 2 gives:

$$\mathbf{180° = \pi}$$

Dividing both sides by π gives:

$$1\ rad = \frac{180°}{\pi}$$

Dividing both sides by $180°$ gives:

$$\frac{\pi}{180°} = 1\ rad$$

So, to convert from radians to degrees:

➔ ***Multiply the angle by*** $\dfrac{180°}{\pi}$

And to convert from degrees to radians:

➔ ***Multiply the angle by*** $\dfrac{\pi}{180°}$

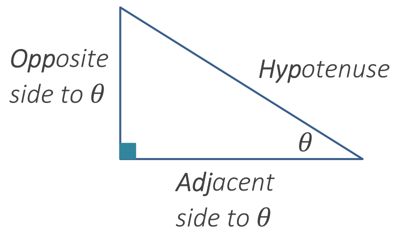
6.2 Trig Ratios of Angles in Standard Position

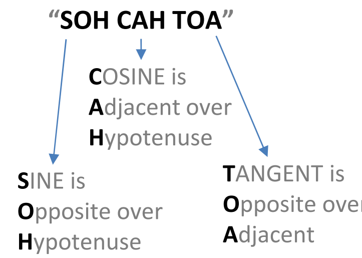
In previous courses we encountered the three primary trigonometric ratios:

The sine ratio: $sin\,\theta = \frac{opp}{hyp}$

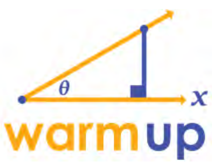
The cosine ratio: $cos\,\theta = \frac{adj}{hyp}$

The tangent ratio: $tan\,\theta = \frac{opp}{adj}$





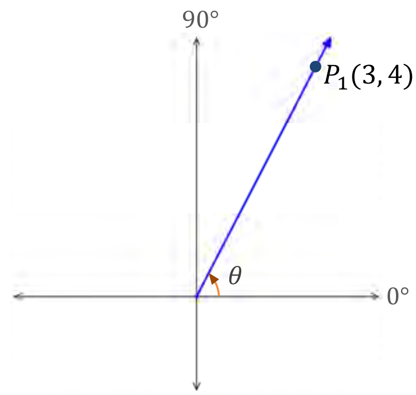
In this chapter we'll study how these primary trigonometric ratios, along with the three reciprocal trigonometric ratios, can be applied to angles in standard position.



Answers for this warm-up are at the bottom of the next page

Consider and angle in standard position θ, that passes through a point $P_1(3, 4)$.

1 ➡ Construct a triangle by drawing a line segment from the point P down to the x-axis. Indicate the right angle with a ■.

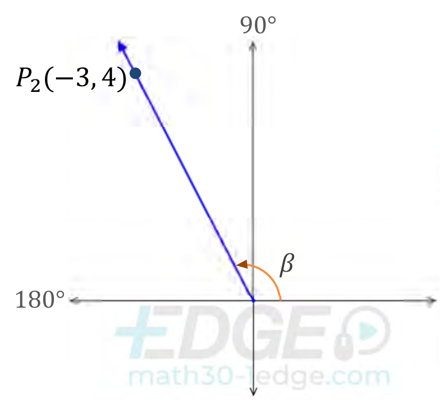


2 ➡ Label the length of each of the sides of the triangle. Use the Pythagorean theorem to determine the length of the hypotenuse.

3 ➡ State the three primary trigonometric ratios of θ; *sine, cosine,* and *tangent*.

For questions 4, 5, and 6, refer to a new angle in standard position β, that passes through $P_2(-3, 4)$

4 ➡ Construct a triangle by drawing a line segment, parallel to the y-axis, connecting P_2 to the x-axis. Indicate the right angle with a ■. *Note that β should be **outside** the triangle!*



5 ➡ Triangles drawn from an angle in standard position can have *negative side lengths*, as determined by the coordinates / quadrants. Label each side length.
Note that the hypotenuse is always positive.

6 ➡ The angle supplementary to β, located inside the triangle, is the **reference angle**. Label this reference angle α. (that is, "alpha")

7 ➡ State the three primary trigonometric ratios of β; *sine, cosine,* and *tangent*.

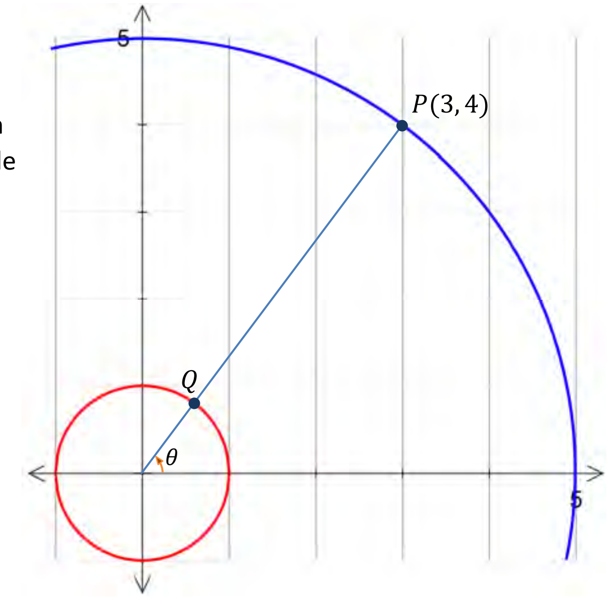
8 ➡ Describe how the trig ratio values for β compare to those for θ.

6.3 The UNIT CIRCLE

warm up

Exploration #1

Both circles in the diagram on the right are centered at the origin. The partially-shown outer circle has a radius of 5, the inner circle a radius of 1.



1 ➡ Construct a triangle by drawing a line segment from the point P down to the x-axis. Indicate the right angle with a ■. Label each side of the triangle.

2 ➡ Use the triangle drawn to determine the values of:

$sin\theta =$ $cos\theta =$

3 ➡ Construct a smaller triangle by drawing a line segment from the point Q down to the x-axis. Use proportions to determine the length of each side, and the coordinates of Q.

4 ➡ Use the smaller triangle to determine the values of: $sin\theta =$ $cos\theta =$

5 ➡ How do the values of $cos\theta$ and $sin\theta$ relate to the sides of the triangle when the hypotenuse is 1?

6 ➡ How do the values of $cos\theta$ and $sin\theta$ relate to the coordinates of a point on the smaller circle?

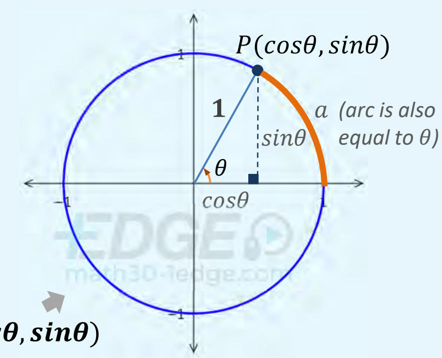
Defining the Unit Circle

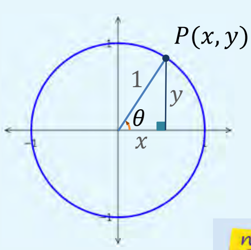
So as we can see from the warm-up above, a circle that's centred at the origin with a radius of one has particularly interesting property. Did you notice it?

The coordinates of the points are the cosine and sine values for the corresponding angle in standard position!

A **unit circle** has its centre at the origin, and a radius of 1.

The equation is $x^2 + y^2 = 1$, which comes from the Pythagorean Theorem.





And once again, here's the good part…

⇦ Since the hypotenuse is 1….

$sin\theta = \frac{y}{1}$ ⟵ opp / ⟵ hyp $\quad cos\theta = \frac{x}{1}$ ⟵ opp / ⟵ hyp

…The coordinates of any point are $(x, y) = (\boldsymbol{cos\theta}, \boldsymbol{sin\theta})$

note: As the arc length, a, is equal to $r\theta$, and $r = 1$, the length of any arc on a unit circle is **equal to** the radian measure of the angle in standard position θ

The unit circle is a useful tool for visualizing the values of $\boldsymbol{sin\theta}$ and $\boldsymbol{cos\theta}$. In the diagram above, ↑ try to visualize $\boldsymbol{P}$ as it "rides up" the circle from $\mathbf{0}$ to $\boldsymbol{\pi/2}$. Can you see how the x-coord "shrinks" … from 1 to 0? *That's what's happening to the value of $cos\theta$!* And how about the $\boldsymbol{y}$-coord. ($\boldsymbol{sin\theta}$), as θ varies from 0° to 90°?

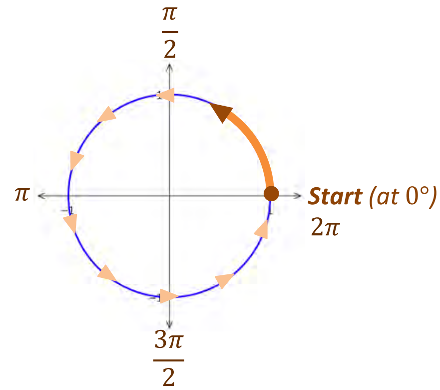
We'll come back to this concept in the next section.

warm up

Investigation 1 - The Graph of $y = sin\ x$

Consider the values of $sin\ \theta$ throughout one rotation on the unit circle. *IE, the y-coordinate*

Through what range of values does it vary?



1 ➡ Complete the table of sine values below:

Angles within one unit circle rotation

θ	$sin\theta$
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$3\pi/4$	
π	
$5\pi/4$	
$3\pi/2$	
$7\pi/4$	
2π	

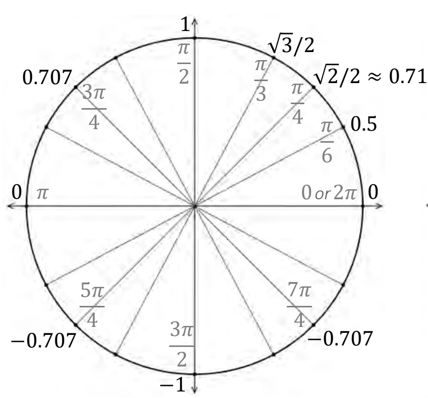
Angles outside $[0, 2\pi]$

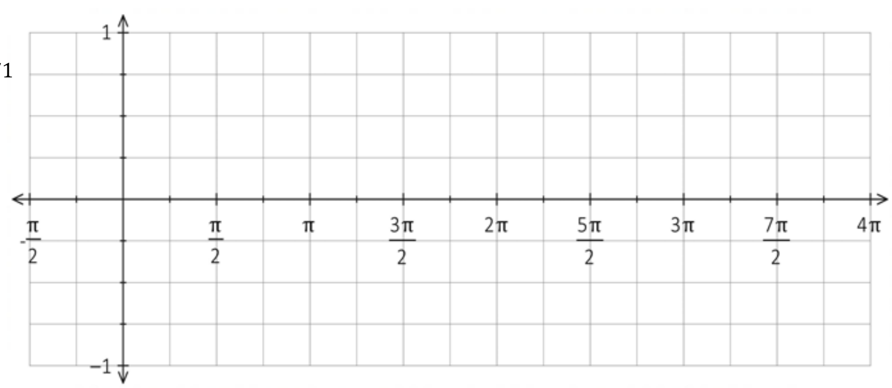
θ	$sin\theta$
$5\pi/2$	
3π	
$7\pi/2$	
4π	
$-\pi/2$	

2 ➡ Complete the table of sine values below:

θ interval	Behavior of $sin\theta$
From 0 to $\pi/2$	*Goes from* ____ *to* ____
From $\pi/2$ to π	*Goes from* ____ *to* ____
From π to $3\pi/2$	*Goes from* ____ *to* ____
From $3\pi/2$ to 2π	*Goes from* ____ *to* ____
From 2π to $5\pi/2$ *principal angle 0 to $\pi/2$*	*Goes from* ____ *to* ____
From $5\pi/2$ to 3π *$\pi/2$ to 3π*	*Goes from* ____ *to* ____

3 ➡ Use the tables above to plot the graph of $y = \sin x$, on the interval $[-\pi/2, 4\pi]$





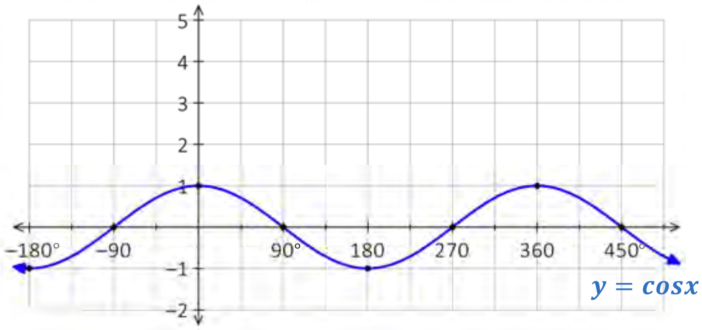
4 ➡ The **PERIOD** of the graph is defined as the horizontal length of one full cycle. State the period.

5 ➡ The **AMPLITUDE** is defined as the vertical distance to the max or the min, from a mid-line through the graph. State the amplitude.

Investigation 1 – **Vertical Displacement**

warm up

1 ➡ The graph of $f(x) = \cos x$ is shown below. Use transformations to sketch the graph of $\boldsymbol{g(x) = 2\cos x + 3}$



Complete the mapping rule to transform each indicated point (•) on the graph of $y=sinx$ *Keep it going!*

$(x, y) \rightarrow$

$(-180°, -1) \rightarrow$

2 ➡ On the graph of $f(x) = cosx$ the x-axis serves as the **median line**, which is a *horizontal line that cuts directly through the middle of a sinusoidal curve*. **Sketch** the dotted median line for your transformed graph $y = g(x)$.

3 ➡ State the range of the resulting function.

4 ➡ Explain how the range of $y = a\cos x + d$ relates to the a and d values.

Vertical Translations – Effect of $\boldsymbol{d}$ in the graph of $\boldsymbol{y = a\sin(bx) + d}$ or $\boldsymbol{y = a\cos(bx) + d}$

So far we've seen how $\boldsymbol{a}$ is a vertical stretch , which affects the **amplitude**, and $\boldsymbol{b}$ represents a horizontal stretch (of $1/b$), which affects the **period**.

Next we'll examine **translations**, which cause the vertical or horizontal shifting of graphs, with no change to the *shape or orientation*.

In the graph of $\boldsymbol{y = asinx + d}$, (or $y = a\cos bx + d$)

- $\boldsymbol{d}$ can be visualized as the distance from the **median line** to the x-axis, and
- $\boldsymbol{a}$ the distance from the median line to the max or min point (the **amplitude**).

➔ *The effect of* $\boldsymbol{d}$ *is a* ***vertical translation****, while* $\boldsymbol{a}$ *is a* ***vertical stretch*** *about the x-axis.*

Therefore, the **range** is $[-a + d\,, a + d]$

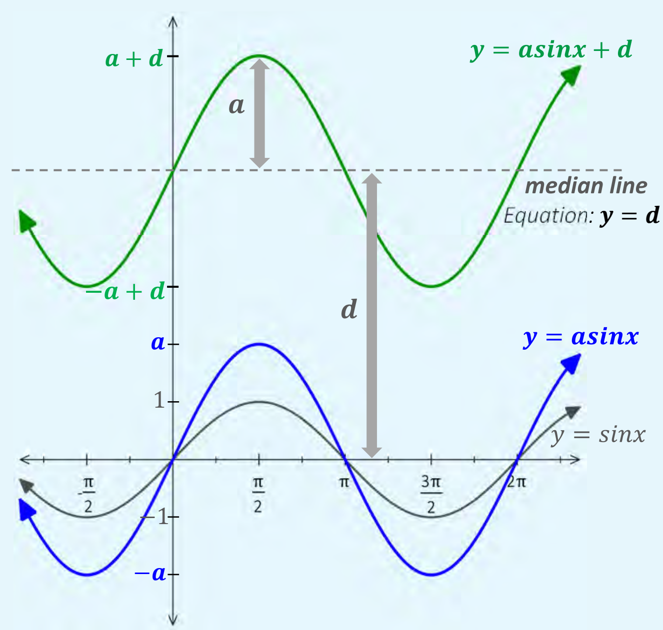
As $\boldsymbol{d}$ represents the median value of the function, it can be found using the formula:

$$\boldsymbol{d} = \frac{max + min}{2}$$

Whereas for $\boldsymbol{a}$ we can use:

$$\boldsymbol{a} = \frac{max - min}{2}$$

Note: By "max" and "min" we refer to y-coordinates!

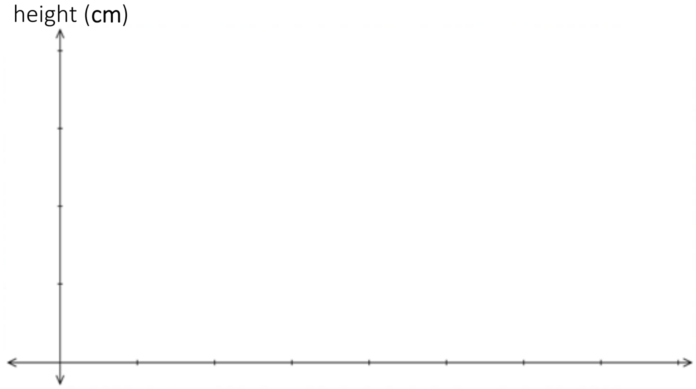


note: Draw your median line so that the distance to the max is the same as the distance to the min!

6.6 Modelling Real-World Applications with Sinusoidal Functions

A rock is stuck in a bicycle tire that has a diameter of 40cm. The tire is rotated so that it completes one rotation every 4 seconds. At $t = 0$ the rock is at the lowest position.

A sinusoidal equation that models the height of the rock (in cm) after t seconds can be written in the from $\boldsymbol{h(t) = a\cos[b(x - c)] + d}$ or $\boldsymbol{h(t) = a\sin[b(x - c)] + d}$.



1 Provide a scale for each axis to sketch a graph showing the height of the rock over two complete rotations.

2 Determine the following sinusoidal function graph characteristics, as they apply to the path of the rock.

Amplitude: Vertical Displacement: Period: "b" value:

3 Using a minimum phase shift, state an equation that models the height of the rock, $h(t)$, as a:

i sine equation, with $a > 0$

ii cosine equation, with $a > 0$

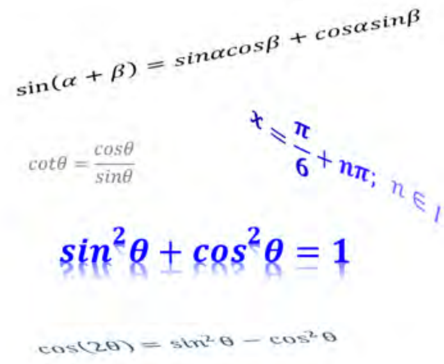
iii cosine equation, with $a < 0$

4 Use one of the equations to determine the height of the nail after 4.5 seconds. (Round to the nearest cm)

Sinusoidal functions can be used to model many real-world scenarios with periodic behavior or wave characteristics. Examples are height involving circular motion (the warm-up above, Ferris wheels), daily temperature or time of sunrise / sunset in a region throughout the year, height of water during tides, etc.

When modelling real-world scenarios we exclusively use **radian mode**.

Chapter 7 TRIGONOMETRIC IDENTITIES and EQUATIONS



7.1 Trigonometric Identities

Welcome to **part II** of Trigonometry! We'll now move into **trigonometric identities**, many basic forms of which are listed on your formula sheet.

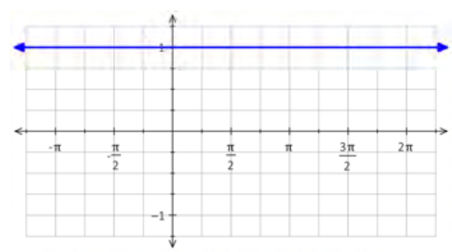
warm up

1 ➡ Use your graphing calculator to sketch each of the following trigonometric functions. Match with the correct graph on the right.

❶ $y = \dfrac{\sin x}{\cos x}$

❷ $y = \dfrac{\cos x}{\sin x}$

❸ $y = \sin^2 x + \cos^2 x$

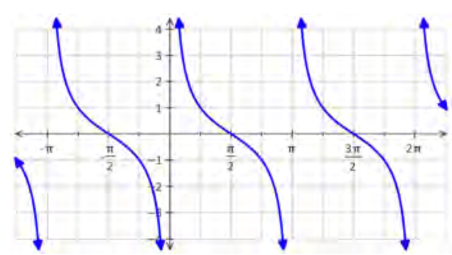


Function # _____

CALC TIP

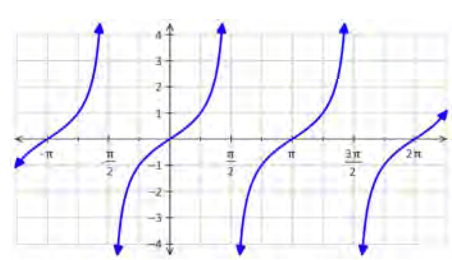
Graph in radian mode,
Match your window settings

```
WINDOW
 Xmin=-5π/4
 Xmax=9π/4
 Xscl=π/4
 Ymin=-4
 Ymax=4
 Yscl=1
 Xres=1
```



Function # _____

2 ➡ Refer to your formula sheet to determine a trigonometric identity that applies to each function graphed above. List each here.



3 ➡ How do the graphs of both sides of an identity relate?

Function # _____

Warm-up is continued on the next page

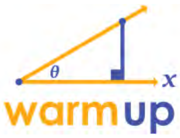
Sum, Difference, and Double-Angle Identities

Investigation 1 – The Sum Formulas

We start with three core intro questions: Given two angles, $\angle\alpha$ and $\angle\beta$, does: $\sin(\alpha+\beta)=\sin\beta+\sin\beta$?

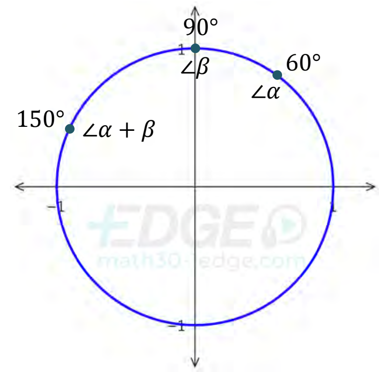
or does $\cos(\alpha+\beta)=\cos\alpha+\cos\beta$?

how about $\tan(\alpha+\beta)=\tan\alpha+\tan\beta$?



These are good questions! Or they are at the least questions. Let's investigate....

1 ➡ Label the coordinates of the 3 points marked on the unit circle. ➔
Given $\angle\alpha=60°$ and $\angle\beta=90°$, evaluate each of the following:



2 ➡ i $\sin(\alpha+\beta)$ ii $\sin\alpha+\sin\beta$

3 ➡ i $\cos(\alpha+\beta)$ ii $\cos\alpha+\cos\beta$

4 ➡ i $\tan(\alpha+\beta)$ ii $\tan\alpha+\tan\beta$

5 ➡ Based on your results, what conclusion can we make regarding the core intro questions above?

The trigonometric ratios of angle sums can be found using the sum identities, first developed by ancient mathematician and astronomer Claudius Ptolemy during the 2nd century AD.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

We can use these formulas without understanding how they are derived. However, we'll refer to these two core identities to develop further formulas in this chapter.

For the curious, do a web search for "proof of the sum and difference identities". And enjoy!

note

When dealing with angle measures, you may have noticed we use a lot of Greek letters!

π and θ (theta) are both from the Greek alphabet, as are the two new variables we'll use in this chapter:

α "alpha"
β "beta"

7.3 Proving Trigonometric Identities

warm up

In section 7.1 we saw how a trigonometric **identity** is an equation that is true for all defined values of the variable in the expressions on both sides. In this section we'll differentiate between methods used to **verify** identities, and those used to **prove** them.

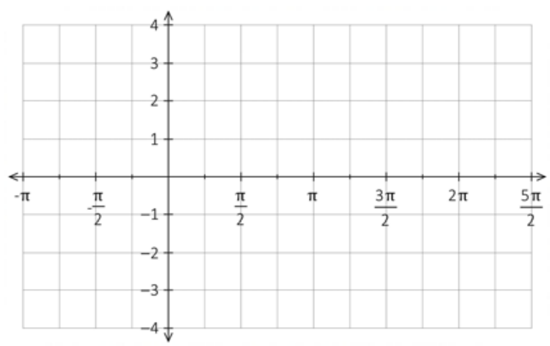
Consider the equation: $\cos\theta + \tan\theta\sin\theta = \sec\theta$

1 ➡ Determine the non-permissible values for this identity.

2 ➡ Numerically verify that $\cos\theta + \tan\theta\sin\theta = \sec\theta$ when $\theta = 30°$, and for any other permissible value θ.

3 ➡ Graphically verify that $\cos\theta + \tan\theta\sin\theta = \sec\theta$ could be an identity. Provide a sketch of your graph here, labeling all relevant characteristics.

Do the graphs verify the potential of an identity? Explain your reasoning.



4 ➡ Use the algebraic process described below, on the left side, to **prove** that $\cos\theta + \tan\theta\sin\theta = \sec\theta$ is an identify.

	L.S.	R.S.
	$\cos\theta + \tan\theta\sin\theta$	$\sec\theta$
Re-write so all functions are in terms of sine and cos.	$= \cos\theta + \left(\quad \right)\sin\theta$	
"Prep" the terms – obtain a common denominator	$= \underline{\qquad} + \underline{\qquad}$	
Combine into a single rational expression	$= \underline{\qquad\qquad}$	
Substitute the numerator using the Pythagorean Identity.	$= \underline{\qquad}$	
Apply the applicable reciprocal identity.		

7.4 Trigonometric Equations

warm up

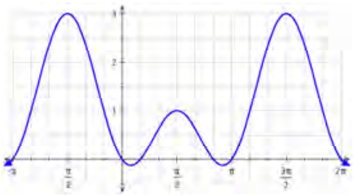
Consider the following trigonometric equations:

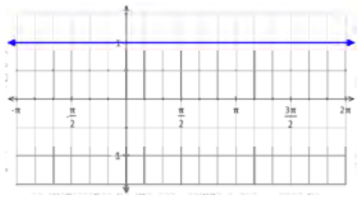
i $\sin^2\theta + \cos^2\theta = 1$ ii $2\sin^2\theta - \sin\theta = 0$ iii $\sin 2\theta + \cos\theta = 0$

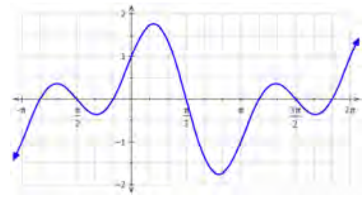
1 ➡ Explain how equation i is different from equations ii and iii.

2 ➡ Below is the graph of the left side of each of the three equations. Use your graphing calculator to match each equation above with the correct graph. Label each.

equation *equation* *equation*







Solutions: ______ *on* $[0, 2\pi)$

______ *on* $[-\pi, \pi)$

Solutions: ______ *on* $[0, 2\pi)$

______ *on* $[-\pi, \pi)$

3 ➡ The graphs on the left and right represent equations whose specific solutions can be found by analyzing the graphs. For those two equations, state the solutions on each indicated domain.

Consider the trigonometric equation: iv $2\sin\theta - 1 = 0$; on $0 \le \theta < 2\pi$

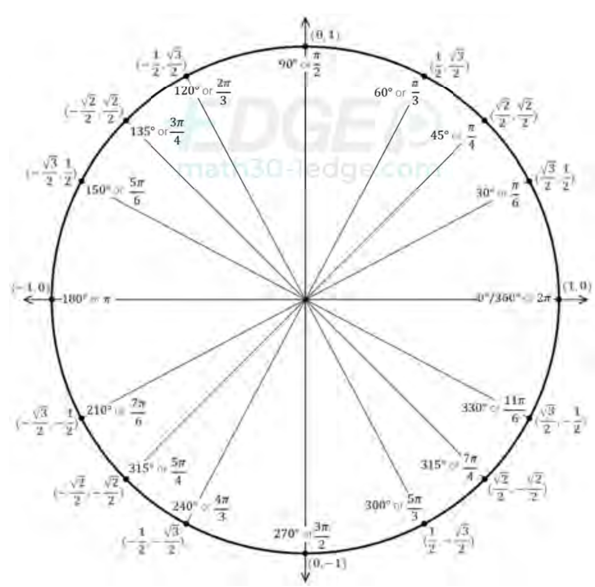
4 ➡ Determine the equation solutions by:

- First isolating the trig term
- Referencing the unit circle

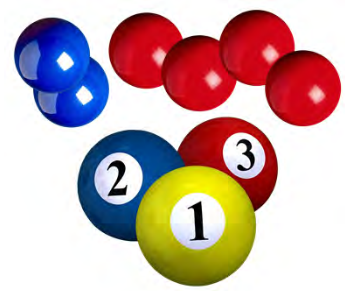
Indicate the solutions on the circle ➔

5 ➡ Use an identity to algebraically solve the following equation, on $[0, 2\pi)$:

iii $\sin 2\theta + \cos\theta = 0$



Chapter 8 PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL THEOREM



8.1 The Fundamental Counting Principal and Permutations

warm up $_nP_r = n!/(n-r)!$

Suppose you are at a restaurant that has a breakfast special, where you get scrambled eggs, a choice of three types of protein, and a choice of two types of toast.

Breakfast Special	Enjoy our ***Scrambled Eggs***, plus:	**Protein** Choose one of: ***Bacon, Sausage,*** *or* ***Ham***	**Toast** Choose one of: ***Wheat*** *or* ***Rye***

1 ➡ Assuming you'll order a complete breakfast, how many different order options do you have? Visually show how you obtained your answer.

2 ➡ The restaurant server informs you that you can also choose between coffee or juice for your beverage. How many different orders consisting of scrambled eggs, a protein, toast, and a beverage are possible?

3 ➡ The restaurant decides to get with the times and offer a plant-based protein option, a tofu patty. How many complete order options are there now (including beverage), with four choices of protein?

The fundamental counting principal (also called the counting rule) is a way to determine the number of outcomes in a problem involving different options at each stage.

If there are a options for the first stage, then b for the second, c for the third (and so on…)

Then the number of possible outcomes is $a \times b \times c \times \ldots$

warm up $_nP_r = n!/(n-r)!$

Warm-Up #1

A family of four; Mom, Dad, and two girls Becky, and Sandra are lining up in a single row for a family photo. M D B S

Dad, always one to throw out challenges, asks his children how many different the family can arrange themselves in a single line.

1 ➡ Determine the number of ways the family members can be arranged in a single line

2 ➡ Determine the number of ways the family members can be arranged if Mom must be on the far left

The photographer suggest they take a few pictures where mom and dad are standing together, with mom on the left and dad on the right. Once again Dad asks how many different ways this can be done.

Sandra, who studies grade 12 math and loves logic problems, asserts that since they are side-by-side in each photograph, Mom and Dad can be counted as one person. M,D B S

3 ➡ Determine the number of ways the family members can be arranged in this fashion, with Mom and Dad together, with mom on the left and dad on the right.

4 ➡ Determine the number of ways the family members can be arranged in this fashion, with Mom and Dad together, on either side of each other.

In this section we'll explore some common types of restrictions in problems involving permutations.

One such restriction is when one (or more) of the objects being arranged *must be in a specific spot*, as with #2 above. In these cases, we can simply **arrange the remaining objects**.

Permutations where Certain Objects are Together

As we saw above, the key to approaching arrangements where some of the objects must grouped together is to count the grouped objects as **one**.

Number of arrangements where some objects *must be grouped together* = # of arrangements, counting grouped objects as one × # ways to arrange the grouped objects among themselves

Worked Example In how many ways can the letters in the word DIPLOMA be arranged if:
(a) The vowels must be together, in alphabetical order (b) The vowels must be together

Solution:

(a) **Count the vowels as one:**

D | P | L | M | AIO

There are 5 letters to arrange

$= 5! \times 1$ ➡ $=$ **120**

Ways to arrange all letters, *counting vowels as one* (5!); Ways to arrange vowels *in alphabetical order* (1)

(b) *This time, vowels can be in any order*

Count the vowels as one:

$= 5! \times 3!$ ➡ $=$ **720**

Ways to arrange all letters, *counting vowels as one* (5!); Ways to arrange vowels among themselves (3!)

8.3 Combinations

warm up $_nP_r = n!/(n-r)!$

Exploration #1

Remember our group of students, from a previous section:

1. **A**bdel
2. **B**arack
3. **C**ecil
4. **D**haria
5. **E**ve
6. **F**rancis
7. **G**arreth

Let's suppose that each of these students are running for student council, where the positions are President, Secretary, and Treasurer.

1 ➡ **Determine the number of ways the student council positions can be filled from this group. Is the order in which the students are elected matter?**

Now let's suppose that after filling the positions above, the four leftover students were **A**bdel, **B**arack, **C**ecil, and **D**haria. From these four students, three student council advisors are needed.

2 ➡ **Does the order in which the three students are selected to fill the advisor groups matter?**

3 ➡ **List each of the possible three-member advisor groups that can be formed from the four leftover students.**
The first is listed for you, using the first letters of their names.

A, B, C

Exploration #2

Above you just listed all of the 3-letter combinations (where order doesn't matter) of **A**, **B**, **C**, and **D**.

Let's compare that to the list of all of the 3-letter permutations of the same letters:

A, B, C	A, B, D	A, C, D	B, C, D
A, C, B	A, D, B	A, D, C	B, D, C
B, A, C	B, A, D	C, A, D	C, B, D
B, C, A	B, D, A	C, D, A	C, D, B
C, A, B	D, A, B	D, A, C	D, B, C
C, B, A	D, B, A	D, C, A	D, C, B

1 ➡ **From this list and that you made above, we can see that:**

From 4 objects, the number of **combinations** of any 3 of them is _______.

From 4 objects, the number of **permutations** of any 3 of them is _______.

2 ➡ **How many times larger is the number of permutations compared to combinations?**
How does this result relate to the list of permutations above?

In a **permutation**, or arrangement of objects, the order is taken into account.

In a **combinations**, or selection of objects, order is not taken into account.

If from n objects, r are selected, The number of COMBINATIONS $= \dfrac{\text{\# of permutations}}{\text{\# of variants for each selection}} = \dfrac{_nP_r}{r!}$

Note that in warm-up 2 we found that the number of PERMS was $r!$ times larger than the number of COMBS.

You may also note that there's a similarity with the way we developed the permutations formula,

$$\text{\# of PERMS} = \frac{\text{\# of ways to arrange all } n \text{ items}}{\text{\# of ways to arrange items } \textit{not wanted}}$$

In both cases we divide by the number of redundancies counted in the numerator.

Warm-Up #1

Pascal's Triangle

For the first three rows of Pascal's Triangle on the right, each value is given by the combs term below. The first value (for row 1) is indicated.

1 ➡ Indicate the values for rows 2 and 3, by evaluating each combs term below.

2 ➡ Follow the patterns to fill out the values for rows 4, 5, and 6. Also indicate the "n" and "r" values for each combs term.

3 ➡ Fill out the values only (without corresponding combs terms) for the 7th and 8th rows.

5 ➡ Use the patterns you've observed state the first three combs' terms for the 25th row of the triangle.

6 ➡ Predict the number of terms there will be at the 25th row.

1 / $_0C_0$ ← 1st row

$_1C_0$ $_1C_1$ ← 2nd row

$_2C_0$ $_2C_1$ $_2C_2$ ← 3rd row

$_3C_0$ $_3C_1$ $_3C_2$ $_3C_3$ ← 4th row

C C C C C ← 5th row

C C C C C C ← 6th row

C C C C C C C

Pascal's Triangle was described by French mathematician Blasé Pascal in 1653. There are many pattens that were used in the tasks above, including how combinations can be used to determine each value.

Warm-Up #2 The Expansion of $(x + y)^n$

We next consider the expansion of the binomial $(x + y)^n$, to various degrees:

You may notice the how the coefficients relate to the patters studied in warm-up #1. Use that and other patterns you notice to:

1 ➡ Fill in the blanks for the missing coefficient values of $(x + y)^4$

2 ➡ Fully expand $(x + y)^5$ ⟶

3 ➡ Predict the number of terms there will be in the expansions of $(x + y)^6$ and $(x + y)^{12}$

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + \_x^3y + \_x^2y^2 + \_xy^3 + y^4$

$(x + y)^5 =$ ➡

4 ➡ Use combinations to predict the first three coefficients in the expansion of $(x + y)^{12}$

5 ➡ **Fill in the blanks:** In the expansion of $(x + y)^n$, there are _____ terms, and the sum of the exponents for each term is ____. The exponents of x go from ____ to ____, while the exponents of y go from ____ to ____.

The combs expressions for first three coefficients in the expansion of $(x + y)^n$ are: ____ (*term 1*), ____ (*term 2*), and ____ (*term 3*)

While for the last term the coefficient is: ____ (*last term*)

6 ➡ In the expansion of $(x + y)^n$, for the exponent of $\boldsymbol{y}$ is "0" for term # ____, it's "1" for term # ____, "2" for term # ____, and in general it is "$\boldsymbol{k}$" for term # ______.